

On the genuine bound states of a non-relativistic particle in a linear finite range potential

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Abstract

We explore the energy spectrum of a non-relativistic particle bound in a linear finite range, attractive potential, envisaged as a quark-confining potential. The intricate transcendental eigenvalue equation is solved numerically to obtain the explicit eigen-energies. The linear potential, which resembles the triangular well, has potential significance in particle physics and exciting applications in electronics.

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1 Introduction

A challenging problem in particle physics in recent years is that of quark confinement. It is presently known that mesons are not elementary particles, but are composed of quarks, as are the nucleons.

In literature, several approximation methods are available relating to quark confinement. The *lattice model* [1] suggests that at large distance between quarks, the interaction increases approximately linearly with separation. The *bag model*, where quarks and gluons are confined in a bag, is not suitable for calculating the hadronic properties of heavy quarks or in computing the energy levels of excited states. *String model*, on the other hand, proposes quark-antiquark pair at the ends of an open string and creation of quark-antiquark pair when the string breaks. In recent years, *potential models* [2] are best justified theoretically to describe heavy quarkonia and seem to be most powerful in calculating the static properties.

Several authors [3 - 6] have addressed the bound states of various kinds of linear potential. Chiu [7] has examined the quarkonium systems with the regulated linear plus Coulomb potential in momentum space. Deloff [8] has used a semi-spectral Chebyshev method for numerically solving integral equations and has applied the same to the quarkonium bound state problem in momentum space.

Rao and Kagali [9 - 11] have investigated extensively the bound states of both spin and spinless particles in a screened Coulomb potential, having linear behaviour near the origin. In the present work, we propose a finite, short-ranged linearly rising potential, envisaged as a quark-confining potential and explore the non-relativistic bound states.

2 The Schrodinger equation with the linear potential

Several attempts have been made to study the meson spectra using the non-relativistic Schrodinger equation with a linear potential. Intuitively, we construct a simple linear rising, finite range potential of the form [12]

$$V(x) = -\frac{V_0}{a} (a - |x|), \quad (1)$$

in which the well depth V_0 and range $2a$ are positive and adjustable parameters. The linear potential with its boundary regions is illustrated in Fig.1 and owing to its shape, this potential could also be called the triangular potential well.

Obviously in regions I and IV, the particle is free and the allowed solutions of the free particle Schrodinger equation are

$$\psi_1(x) = C_1 e^{\alpha x} \quad -\infty < x \leq -a \quad (2)$$

$$\psi_4(x) = C_6 e^{-\alpha x} \quad a \leq x < \infty, \quad (3)$$

consistent with the requirement $\psi(x)$ vanishes as $|x| \rightarrow \infty$.

Here $\alpha^2 = -\frac{2mE}{\hbar^2}$ is implied. Since $E < 0$ for bound states, α is positive. To discuss the nature of the solution within the potential region, $-a < x < a$, we insert the potential described in Eqn.(1) in the celebrated Schrodinger equation and obtain

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \left[E + V_0 \left(1 - \frac{|x|}{a} \right) \right] \psi(x) = 0. \quad (4)$$

Setting $\frac{x}{a} = y$ and defining $\bar{E} = \frac{E}{\hbar^2/2ma^2}$ and $\bar{V}_0 = \frac{V_0}{\hbar^2/2ma^2}$ we obtain the dimensionless form of the Schrodinger equation

$$\frac{d^2\psi(y)}{dy^2} + \left[\bar{E} + \bar{V}_0(1-y) \right] \psi(y) = 0, \quad (5)$$

which may further be written as

$$\frac{d^2\psi}{dy^2} - Ay\psi + B\psi = 0. \quad (6)$$

The constants $A = \bar{V}_0$ and $B = \bar{E} + \bar{V}_0$ are also dimensionless. Introducing an auxiliary function

$$w = A^{\frac{1}{3}} \left(y - \frac{B}{A} \right) \quad (7)$$

yeilds

$$\frac{d^2\psi}{dw^2} - w\psi = 0. \quad (8)$$

The solutions of this differential equation are the well-known Airy functions $Ai(w)$ and $Bi(w)$ [13], having oscillatory and damping nature.

The Eigenvalue Equation

The admissible solutions in the four regions, consistent with physical reality, are

$$\psi_1(x) = C_1 e^{\alpha x} \quad -\infty < x \leq -a \quad (9)$$

$$\psi_2(x) = C_2 Ai(w) + C_3 Bi(w) \quad -a \leq x \leq 0 \quad (10)$$

$$\psi_3(x) = C_4 Ai(-w) + C_5 Bi(-w) \quad 0 \leq x \leq a \quad (11)$$

$$\psi_4(x) = C_6 e^{-\alpha x} \quad a \leq x < \infty \quad (12)$$

where C_1 to C_6 are the normalisation constants. Imposing on the solutions in equations (9) to (12) the requirements that ψ and $\frac{d\psi}{dx}$ be continuous at the origin and also at the potential boundaries ($x = \pm a$) leads to the eigenvalue eqnuation.

At $x = -a$, $\psi_1(x) = \psi_2(x)$ and $\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$

This leads to

$$C_1 e^{-\alpha a} = C_2 \text{Ai}(w_1) + C_3 \text{Bi}(w_1) \quad (13)$$

$$\alpha C_1 e^{-\alpha a} = \frac{A^{\frac{1}{3}}}{a} [C_2 \text{Ai}'(w_1) + C_3 \text{Bi}'(w_1)] \quad (14)$$

where

$$w_1 = A^{\frac{1}{3}} \left(-1 - \frac{B}{A} \right) \quad (15)$$

On simplification, we obtain

$$\alpha = \frac{A^{\frac{1}{3}}}{a} \left[\frac{P \text{Ai}'(w_1) + \text{Bi}'(w_1)}{P \text{Ai}(w_1) + \text{Bi}(w_1)} \right] \quad (16)$$

where $P = \frac{C_2}{C_3}$. Similarly, the continuity condition at $x = 0$ demands

$$\psi_2(x) = \psi_3(x) \text{ and } \frac{d\psi_2}{dx} = \frac{d\psi_3}{dx},$$

from which we obtain

$$C_2 \text{Ai}(w_0) + C_3 \text{Bi}(w_0) = C_4 \text{Ai}(-w_0) + C_5 \text{Bi}(-w_0) \quad (17)$$

$$C_2 \text{Ai}'(w_0) + C_3 \text{Bi}'(w_0) = C_4 \text{Ai}'(-w_0) + C_5 \text{Bi}'(-w_0) \quad (18)$$

with

$$w_0 = A^{\frac{1}{3}} \left(\frac{-B}{A} \right). \quad (19)$$

As before,

$$\frac{P \text{Ai}'(w_0) + \text{Bi}'(w_0)}{P \text{Ai}(w_0) + \text{Bi}(w_0)} = \frac{Q \text{Ai}'(-w_0) + \text{Bi}'(-w_0)}{Q \text{Ai}(-w_0) + \text{Bi}(-w_0)} \quad (20)$$

where $Q = \frac{C_4}{C_5}$ is another constant. Adapting similar procedure at the boundary $x = +a$, demanding $\psi_3(x) = \psi_4(x)$ and $\frac{d\psi_3}{dx} = \frac{d\psi_4}{dx}$ one would on similar grounds obtain

$$-\alpha = \frac{A^{\frac{1}{3}}}{a} \left[\frac{Q \text{Ai}'(-w_2) + \text{Bi}'(-w_2)}{Q \text{Ai}(-w_2) + \text{Bi}(-w_2)} \right] \quad (21)$$

with

$$w_2 = A^{\frac{1}{3}} \left(1 - \frac{B}{A} \right). \quad (22)$$

It is worthwhile mentioning that the arguments of the Airy function w_0 , w_1 , and w_2 are dependent both on the energy as well as the potential and are related by the simple equation

$$w_0 = \frac{w_1 + w_2}{2}. \quad (23)$$

It is straightforward to check that

$$P = \frac{\beta \text{Bi}(w_1) - A^{\frac{1}{3}} \text{Bi}'(w_1)}{A^{\frac{1}{3}} \text{Ai}'(w_1) - \beta \text{Ai}(w_1)}, \quad (24)$$

and

$$Q = - \left[\frac{\beta \text{Bi}(-w_2) + A^{\frac{1}{3}} \text{Bi}'(-w_2)}{\beta \text{Ai}(-w_2) + A^{\frac{1}{3}} \text{Ai}'(-w_2)} \right], \quad (25)$$

where $\beta = \alpha a$ is implied.

Formally on eliminating P and Q in Eqn.(20) we obtain the eigenvalue equation as

$$\begin{aligned} & \left[\frac{\{\beta \text{Bi}(\omega_1) - A^{\frac{1}{3}} \text{Bi}'(\omega_1)\} \text{Ai}'(\omega_0) + \{A^{\frac{1}{3}} \text{Ai}'(\omega_1) - \beta \text{Ai}(\omega_1)\} \text{Bi}'(\omega_0)}{\{\beta \text{Bi}(\omega_1) - A^{\frac{1}{3}} \text{Bi}'(\omega_1)\} \text{Ai}(\omega_0) + \{A^{\frac{1}{3}} \text{Ai}'(\omega_1) - \beta \text{Ai}(\omega_1)\} \text{Bi}(\omega_0)} \right] = \\ & \left[\frac{\{\beta \text{Bi}(-\omega_2) + A^{\frac{1}{3}} \text{Bi}'(-\omega_2)\} \text{Ai}'(-\omega_0) - \{A^{\frac{1}{3}} \text{Ai}'(-\omega_2) + \beta \text{Ai}(-\omega_2)\} \text{Bi}'(-\omega_0)}{\{\beta \text{Bi}(-\omega_2) + A^{\frac{1}{3}} \text{Bi}'(-\omega_2)\} \text{Ai}(-\omega_0) - \{A^{\frac{1}{3}} \text{Ai}'(-\omega_2) + \beta \text{Ai}(-\omega_2)\} \text{Bi}(-\omega_0)} \right] \end{aligned} \quad (26)$$

This intricate and fairly complicated transcendental eigenvalue equation involving the Airy function and its derivatives is solved both graphically and numerically using Mathematica[14]. The real roots, which correspond to the eigenenergies, are listed in Table 1 for a typical value of the range parameter(a). Energy (E) and well-depth (V_0) are both expresed in units of $\frac{\hbar^2}{2ma^2}$.

3 Results and Discussion

One of the distinctive characteristics of quantum mechanics, in contrast to classical mechanics, is the existence of bound states corresponding to discrete energy levels. It is well-known in quantum mechanics that bound states exist for all attractive potentials, the exact number depending on the specific form of the potential and the dimensionality of the space.

More specifically, as is seen from the spectrum of energies listed in Table 1, for a finite range of the potential, deeper wells admit excited state energies, consistent with the wisdom of quantum mechanics. Such studies, apart from being pedagogical in nature, are potentially exciting and significant as it is concerned with quark confinement.

Quantum chromodynamics, which governs the quark-antiquark interaction is widely accepted as a good theory of strongly interacting particles. One can explore the hadronic properties by investigating the bound states of quarks. Our investigation concerning the linear potential is seeming interesting and can be regarded as a model to describe the quarkonia.

Further, the linear potential well or in other words, the triangular well has potential applications in electronics. Interestingly, in many semiconductor devices, it is believed that electrons are confined in almost triangular quantum wells [15]. Examples of such devices are Si MOSFETs (Metal Oxide Semiconductor Field Effect Transistors) widely used in digital applications and GaAs/AlGaAs MODFETs (Modulation Doped Field Effect Transistors) used for high speed applications. The bound states of the linear potential is a subject of renewed interest and intensive research and we have extended the study of this naive potential to the relativistic domain, which will be reported shortly.

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Table 1**Eigenenergies of a non-relativistic particle in a linear potential**

$$(a = 1\lambda)$$

\bar{V}_0	\bar{E}_0	\bar{E}_1	\bar{E}_2
0.01	- 0.0000976		
0.05	- 0.0022242		
0.10	- 0.0080220		
0.20	- 0.0268461		
0.30	- 0.0519188		
0.40	- 0.0808526		
0.50	- 0.1122377		
0.60	- 0.1451770		
0.70	- 0.1790694		
0.80	- 0.2134967		
0.90	- 0.2480162		
1.0	- 0.2828516		
2.0	- 0.6121732		
3.0	- 0.9072505		
4.0	- 1.2056918		
4.28	- 1.2984370	- 0.0000991	
5.0	- 1.5770587	- 0.1653382	
10.0	- 5.8335771	- 1.3641773	
20.0	-16.4420518	- 2.3126501	
20.62	-17.0607820	- 2.3662130	-0.0000343
25.0	-21.2732190	- 3.6112921	-2.4865189
30.0	-25.7497380	- 9.0650370	-2.9501500
35.0	-29.8836000	-15.0604510	-3.2815680
40.0	-33.7181130	-21.1744464	-3.5904940